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Free vibrations of a mass grounded by linear and nonlinear springs in series

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Abstract

In many technical applications spring-like flexible elements or real springs connected in series are used. The operation range of these components determines whether the system behaviour has a linear or nonlinear characteristic. In the relevant literature, there exists the knowledge how the equivalent spring is obtained for linear springs connected serially. However, some cases occur in which one linear and one nonlinear spring arranged in series are used. In such cases, it is not possible to define an equivalent spring rate. This study is concerned with such a system that consists of a mass grounded two springs, one of them linear and the other nonlinear. Two methods are developed to analyze the dynamic behaviour of system. One method makes use of a set of differential-algebraic equations (DAE in short). The other is based on getting a single equation of motion using relative displacement variables. For the second method, analytical solutions are also obtained by means of the Lindstedt and the harmonic balance techniques. It is observed, that numerical and analytical solutions found for both methods are in very good agreement when $v_0 \leqslant 1$ and $0.1 \leqslant \xi \leqslant 10$ where v_0 is the initial deflection of nonlinear spring and ξ is the ratio of linear portion coefficient of the nonlinear spring to that of the linear spring.

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1. Introduction

A mechanical system is said to be linear or nonlinear according to the type of differential equations of motion. The linearity or nonlinearity of a conservative system is determined essentially by the algebraic relationship between restoring forces and displacement/deflections. This property of systems is, in fact, a concept that is related to their range of operation. The small and large amplitude vibrations of a mathematical pendulum form a very good example to demonstrate the basic nuance of linearity and nonlinearity. In discrete models, flexible components producing restoring forces are represented by springs, which have linear or nonlinear characteristics again depending upon their amount of deflection. When there are linear springs in a system, which are connected with each other in series or parallel, they are replaced with their equivalents as is explained in standard textbooks on vibration [1,2]. Sometimes, one of the springs in parallel may be linear while the other is nonlinear. This case results in an equivalent, nonlinear spring, which has a larger coefficient for its linear portion. On the other hand, if a linear spring is connected with a nonlinear one serially, the matter of obtaining an equivalent spring becomes more complicated. In such a case, a possible and practical way could be to fit a low-order polynomial to the force–deflection curve of the combined spring. However, a theoretical approach could also be developed for this problem as is done in the present paper. There exists a vast literature on discrete systems including either linear or nonlinear springs/restoring forces. A literature review can be found in Ref. [3] and especially in Ref. [4]. However, to the authors' knowledge, one does not encounter publications on mechanical systems with single-degree-of-freedom containing flexible component consisting of the combination of one linear and one nonlinear spring in series, which sometimes occur in technical applications. This paper aims to develop methods for analyzing such systems. Two alternative methods are presented. One of them is based on the main idea that the equation of motion is transformed into a set of differential-algebraic equations (henceforth, DAE in short) by introducing intermediate variables when necessary. The second is to get a single equation of motion by means of relative displacements similar to the first method, and to solve this equation by analytical or numerical techniques.

2. Equations of motion

2.1. System with linear springs in series

To simply show how the set of DAE associated with a system having springs connected in series is obtained, one will start with a system including two linear springs in series, then the system with a mixed, serially connected springs will be treated. The word "mixed" here is used to imply that one of the springs in series is linear and the other nonlinear.

In Fig. 1, a mechanical system is shown, which has a mass m grounded by linear springs in serial. In this figure, k_1 and k_2 are linear spring coefficients of stiffness, as y_1 and y_2 denote the absolute displacements of the connection point of two spring, and the mass m, respectively. Since the deflection of equivalent spring must be equal to the summation of individual deflections of the two linear springs, the equivalent stiffness will be $k_1k_2/(k_1+k_2)$ as can be found in standard textbooks [1,2]. The deflection of the spring k_1 is y_1, and that of the spring k_2 is (y_2-y_1), hence

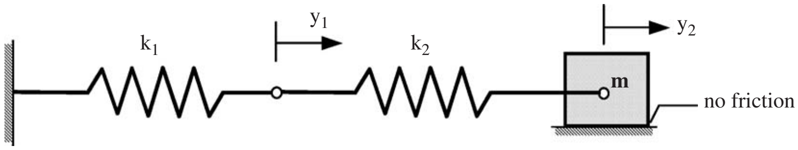

Fig. 1. System having a mass grounded by two linear springs connected in series.

the displacement of the mass m is $y_1 + (y_2 - y_1) = y_2$. Then, the equation of motion of the mass m is as follows:

$$m\ddot{y}_2 + k_{eq} y_2 = 0, \tag{1}$$

where $k_{eq} = k_1 k_2/(k_1 + k_2)$ and the dots over letters denote the derivation with respect to time.

Now, a different way leading to the same result will be followed. The equation of motion of the system in Fig. 1 will be derived by using system's Lagrangian in which y_1 along with y_2 appears as a time-dependent variable. The potential and kinetic energies of such a system are found, respectively, as

$$V = \tfrac{1}{2}k_1 y_1^2 + \tfrac{1}{2}k_2(y_2 - y_1)^2, \tag{2a}$$

$$T = \tfrac{1}{2}m\dot{y}_2^2. \tag{2b}$$

For a conservative system, the Lagrangian of system is defined as

$$L = T - V. \tag{3}$$

Hence, the equations of motions of system will be obtained from

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{y}_i}\right) - \frac{\partial L}{\partial y_i} = 0, \quad i = 1, 2. \tag{4}$$

The equations of motion given by Eq. (4) can be written in the explicit form as follows:

$$k_1 y_1 - k_2(y_2 - y_1) = 0, \tag{5a}$$

$$m\ddot{y}_2 + k_2(y_2 - y_1) = 0. \tag{5b}$$

It is obvious that Eq. (5a) is, in fact, a constraint equation that relates y_1 to y_2 or vice versa. From Eq. (5a), the following relationship can be directly written

$$y_1 = \frac{k_2}{k_1 + k_2} y_2. \tag{6}$$

If Eq. (6) is substituted into Eq. (5b), after some arithmetical manipulations, the equations of motion is obtained as

$$m\ddot{y}_2 + \frac{k_1 k_2}{k_1 + k_2} y_2 = 0 \tag{7}$$

which is actually the same as Eq. (1). However, it should be noted, Eqs. (5) are a set of DAE of index 1 [5]. At this point, one will try to derive the equations of motion in the manner similar to what is just mentioned, differing from the former with the use of Lagrange multiplier. Starting from that two linear springs are subjected to the same tension/compression force, one can write

$$k_1 y_1 = k_2(y_2 - y_1) \tag{8a}$$

or

$$f(y_1, y_2) = k_1 y_1 - k_2(y_2 - y_1) = 0, \tag{8b}$$

where $f(y_1, y_2)$ is a constraint on the system. The potential and kinetic energies do not change, and Eqs. (2a,b) are still valid. However, in the way we follow, the extended Lagrangian of system will be utilized, which is given by

$$L^* = L + \lambda f, \tag{9}$$

where L^* and λ are the extended Lagrangian and the Lagrange multiplier associated with the problem, respectively. The extended Lagrangian can be given explicitly as follows

$$L^* = \tfrac{1}{2}m\dot{y}_2^2 - \tfrac{1}{2}k_1 y_1^2 - \tfrac{1}{2}k_2(y_2 - y_1)^2 + \lambda(k_1 + k_2)y_1 - \lambda k_2 y_2. \tag{10}$$

Now, the number of dependent variables is three, and Eqs. (4) become

$$\frac{\mathrm{d}}{\mathrm{dt}}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L^*}{\partial q_i} = 0, \quad i = 1, 2, 3, \tag{11}$$

where

$$\begin{aligned} q_1 &= y_1, \\ q_2 &= y_2, \\ q_3 &= \lambda. \end{aligned} \tag{12}$$

Hence, the Lagrange equations of the system are found as

$$\lambda(k_1 + k_2) + (k_1 + k_2)y_1 - k_2 y_2 = 0, \tag{13a}$$

$$m\ddot{y}_2 + k_2(y_2 - y_1) - \lambda k_2 = 0, \tag{13b}$$

$$k_1 y_1 - k_2(y_2 - y_1) = 0. \tag{13c}$$

The set of Eqs. (13) constitutes a differential-algebraic system of index 1 similar to Eqs. (5). Considering Eq. (13) one easily concludes that λ must be always zero as is confirmed by numerical solution presented in the following part of the study.

2.2. System with linear and nonlinear springs in series

In this section, the system shown in Fig. 2 will be studied. Note that the second spring is described by two parameters k_2 and β since it has a hardening/softening cubic nonlinear characteristic, i.e., there is the following relationship between the deflection of this spring and the force acting upon it:

$$F_2 = k_2 x + \beta x^3 = k_2 x + \varepsilon k_2 x^3, \tag{14}$$

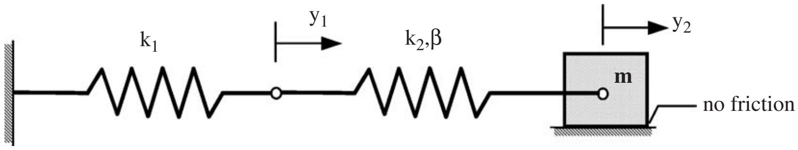

Fig. 2. System with linear and nonlinear springs in series.

where k_2 and β are the coefficients associated with the linear and nonlinear portions of spring force, and ε is defined as

$$\varepsilon = \frac{\beta}{k_2}. \tag{15}$$

According to the notation used in Fig. 2, x is the net deflection of nonlinear spring and defined as

$$x = y_2 - y_1.$$

The case of $\beta > 0$ corresponds to a hardening spring while a negative β indicates a softening one. Here, it is assumed that $\beta > 0$, $\varepsilon < 1$ and ε will be employed as a perturbation or book keeping parameter. The equations of motion of the system in Fig. 2 can easily be obtained as follows:

$$k_1 y_1 - k_2(y_2 - y_1) - \varepsilon k_2 (y_2 - y_1)^3 = 0, \tag{16a}$$

$$m\ddot{y}_2 + k_2(y_2 - y_1) + \varepsilon k_2 (y_2 - y_1)^3 = 0. \tag{16b}$$

Again, one encounters a set of DAEs of index 1, which can be solved numerically by using any ODE-solver like any version of the Runge–Kutta algorithms.

Let the new (intermediate) variables u and v be defined as follows:

$$y_1 := u, \tag{17a}$$

$$y_2 - y_1 := v. \tag{17b}$$

Then, Eq. (16) can be put into a different form

$$k_1 u - k_2 v - \varepsilon k_2 v^3 = 0, \tag{18a}$$

$$m(\ddot{u} + \ddot{v}) + k_2 v + \varepsilon k_2 v^3 = 0. \tag{18b}$$

Solving Eq. (18a) for u yields

$$u = \xi v + \varepsilon \xi v^3, \tag{19}$$

where

$$\xi = \frac{k_2}{k_1}. \tag{20}$$

If Eq. (19) is differentiated twice with respect to time and substituted into Eq. (18b) one finds

$$m(1 + \xi + 3\varepsilon\xi v^2)\ddot{v} + 6m\varepsilon\xi v\dot{v}^2 + k_2 v + \varepsilon k_2 v^3 = 0, \tag{21}$$

where the dots over letters show time derivations.

The problem of solving Eqs. (16) is reduced to solving Eq. (21). It is interesting to observe that a term proportional to velocity squared appears, suggesting that the system contains a dissipative element although this is not the case. Eq. (21) can be considered as a kind of the Duffing equation whose mass and linear spring coefficients are time dependent. If one defines

$$M(t) := m(1 + \xi + 3\varepsilon\xi v^2),$$

$$K(t) := k_2 + 6m\varepsilon\xi\dot{v}^2 \tag{22}$$

then Eq. (21) can be written as

$$M(t)\ddot{v} + K(t)v + \varepsilon k_2 v^3 = 0 \tag{23}$$

since both of the coefficients defined by Eq. (22) are always positive, no matter whether v and $\dot{v}$ are positive or not. Consequently, the solution of Eq. (23) may have quasi-harmonic or periodic motion. The existence of such a motion is immediately concluded for the reason that the system under study is conservative, that is, it does not include any dissipating element such as dry friction or any other.

Once the variable v is found by solving Eq. (21) analytically or numerically, the u values can easily be obtained from Eq. (19). Then, making inverse transformations by Eqs. (17), the original variables y_1 and y_2 are reached. The Lagrangian equations of motion can be rederived using the newly introduced variables u and v, along with a Lagrange multiplier. In this case, the extended Lagrangian of the system becomes

$$L^* = \tfrac{1}{2}m(\dot{u} + \dot{v})^2 - \tfrac{1}{2}k_1u^2 - \tfrac{1}{2}k_2v^2 - \tfrac{1}{4}\varepsilon k_2 v^4 + \lambda(k_1 u - k_2 v - \varepsilon k_2 v^2). \tag{24}$$

The equations of motion are obtained as follows:

$$m(\ddot{u} + \ddot{v}) + k_1 u - \lambda k_1 = 0,$$

$$m(\ddot{u} + \ddot{v}) + k_2 v + \varepsilon k_2 v^3 + \lambda k_2 + 2\lambda\varepsilon k_2 v^2 = 0,$$

$$k_2 v + k_2\varepsilon v^2 - k_1 u = 0. \tag{25a–c}$$

This time, Eqs. (25) represent a system of DAEs of index 3 because it needs to be differentiated three times with respect to time in order to get a set of ODEs. Making some arrangements, however, it is possible to reduce this system to one of index 1. To this end Eq. (25c) is differentiated twice with respect to time and one obtains

$$k_1\ddot{u} - (1 + 3\varepsilon v^2)k_2\ddot{v} - 6\varepsilon k_2 v\dot{v}^2 = 0. \tag{26}$$

If Eq. (25b) is subtracted from Eq. (25a) the following is found:

$$k_1(u + \lambda) - k_2(v - \lambda) - \varepsilon k_2(v^3 - 3\lambda v^2) = 0. \tag{27}$$

After that Eq. (25b) is solved for $\ddot{u}$ and one finds

$$\ddot{u} = -\ddot{v} - \frac{k_1}{m}(u + \lambda). \tag{28}$$

If Eq. (28) is substituted into Eq. (26) and some arrangements are made, one arrives at

$$\ddot{v}[k_1 + (1 + 3\varepsilon v^2)k_2] + 6\varepsilon k_2 v\dot{v}^2 + \frac{k_1^2}{m}(u + \lambda) = 0. \tag{29}$$

Consequently Eqs. (25) are replaced with the following set of equations of index 1:

$$\begin{aligned} &\ddot{v}[k_1 + (1 + 3\varepsilon v^2)k_2] + 6\varepsilon k_2 v\dot{v}^2 + \frac{k_1^2}{m}(u + \lambda) = 0, \\ &m\ddot{u} + m\ddot{v} + k_2(v - \lambda) + \varepsilon k_2(v^3 - 3\lambda v^2) = 0, \\ &k_1(u + \lambda) - k_2(v - \lambda) - \varepsilon k_2(v^3 - 3\lambda v^2) = 0. \end{aligned} \tag{30}$$

While it is possible to solve both Eqs. (21) and (30) numerically, it was attempted to find analytical solution for Eq. (21) to gain a better insight into the system behaviour. The following section is devoted to the presentation of these approximate analytical solutions.

3. Analytical solutions of equation of motion

To obtain an analytical solution to Eq. (21), two different approaches were employed: The Lindstedt method and the harmonic balance method. Before giving the analytical procedures, saying a few words about the initial conditions will be useful. Eq. (21) is an ordinary differential equation in v. The analytical solutions to be presented in this section will also be in terms of v. Therefore, it seems meaningful to give the initial conditions in v. It should be emphasized that any initial condition for v leads to different initial values of y_1 and y_2. For a general initial condition $v(0) = v_0$, one finds

$$\begin{aligned} u(0) = y_1(0) &= \xi v_0(1 + \varepsilon v_0^2), \\ y_2(0) = u(0) + v(0) &= (1 + \xi + \xi\varepsilon v_0^2)v_0. \end{aligned} \tag{31}$$

Eq. (31) relates the initial amount of relative variable v to those of the original ones. In this study, it is assumed that $\dot{y}_1(0) = \dot{y}_2(0) = 0 \Rightarrow \dot{v}(0) = \dot{v}_0 = 0$ for all solutions, either the analytical or the numerical.

3.1. The Lindstedt solution

Eq. (21) can be written as

$$\left(1 + 3\varepsilon\frac{\xi}{1+\xi}v^2\right)\ddot{v} + 6\varepsilon\frac{\xi}{1+\xi}v\dot{v}^2 + \omega_e^2 v + \varepsilon\omega_e^2 v^3 = 0, \tag{32}$$

where

$$\omega_e^2 = \frac{k_2}{m(1+\xi)}. \tag{33}$$

Here, both the solution of v and natural frequency of the system that depends on the amplitude of motion are approximated by two perturbation series to the second-order terms in ε as follows [3]:

$$v(t) = v_0(t) + \varepsilon v_1(t) + \varepsilon^2 v_2(t), \tag{34}$$

$$\omega^2 = \omega_e^2 - \varepsilon\omega_1(A) - \varepsilon^2\omega_2(A), \tag{35}$$

where A is the amplitude of vibration. This approach, of course, is based on the assumption that a periodic motion occurs. Substituting Eqs. (34) and (35) and the time derivatives of $v(t)$ including the terms up to the second order into Eq. (32), and equating the coefficients of the ε-terms of same power yields the following group of equations:

$$\ddot{v}_0 + \omega^2 v_0 = 0,$$

$$\ddot{v}_1 + \omega^2 v_1 = -\frac{3\xi}{1+\xi} v_0^2 \ddot{v}_0 - \frac{6\xi}{1+\xi} v_0 \dot{v}_0^2 + \omega_1 v_0 - \omega^2 v_0^3,$$

$$\begin{aligned}\ddot{v}_2 + \omega^2 v_2 = & -\frac{3\xi}{1+\xi} v_0^2 \ddot{v}_1 - \frac{6\xi}{1+\xi} v_0 v_1 \ddot{v}_0 - \frac{6\xi}{1+\xi} v_1 \dot{v}_0^2 - \frac{12\xi}{1+\xi} \dot{v}_0 \dot{v}_1 v_0 \\ & + \omega_1 v_1 + \omega_2 v_0 + \omega_1 v_0^3 - 3\omega^2 v_0^2 v_1. \end{aligned} \tag{36}$$

The solution of the first of Eqs. (36) is $A_0 \cos(\omega t + \psi_0)$. The remaining ones have homogenous solutions of the same form as the first has. These solutions are not taken into consideration. Following the conventional procedure of the Lindstedt method one obtains

$$\omega^2 = \omega_e^2[1 - \tfrac{3}{4}A_0^2(z-1)\varepsilon + \tfrac{3}{128}A_0^4\varepsilon^2(33z^2 - 34z + 1)], \tag{37a}$$

$$\begin{aligned} v = {} & A_0 \cos(\omega t + \psi_0) - \varepsilon \frac{A_0^3}{32}(9z - 1)\cos(3\omega t + 3\psi_0) \\ & + \varepsilon^2 \left[\frac{3A_0^5}{1024}(153z^2 - 18z - 7)\cos(3\omega t + 3\psi_0) \right. \\ & \left. + \frac{A_0^5}{1024}(225z - 34z + 1)\cos(5\omega t + 5\psi_0) \right], \end{aligned} \tag{37b}$$

where

$$z = \frac{\xi}{1+\xi}. \tag{38}$$

The unknowns A_0 and ψ_0 are found from the initial conditions for Eq. (32).

3.2. Solution by the harmonic balance technique

The main idea behind this method is that any periodic solution of Eq. (32) can be approximated by a truncated Fourier series as follows [4]:

$$v = \sum_{m=0}^{M} A_m \cos(m\omega t + m\psi_0). \tag{39}$$

In this study, we take $M = 3$ and $M = 5$ corresponding to different series with four and six terms, respectively, in order to examine the effect of approximation order on the solution accuracy. The results found for $M = 5$ were observed to be more accurate and consistent with the numerical solution. After substituting Eq. (39) into Eq. (32) for $M = 3$ and then for $M = 5$ and making some tedious manipulations, one reaches the following results for four-term harmonic approximation:

$$A_0 = A_2 = 0, \tag{40a}$$

$$A_3 = -\varepsilon\frac{A_1^3}{32}(9z-1) - \frac{3}{16}\varepsilon^2 A_1^5 z, \tag{40b}$$

$$\omega^2 = \omega_e^2[1 - \tfrac{3}{4}\varepsilon A_1^2(z-1) + \tfrac{9}{16}\varepsilon^2 A_1^4(z^2 - z)], \tag{40c}$$

$$v = A_1\cos(\omega t + \psi_0) - \left[\varepsilon\frac{A_1^3}{32}(9z-1) + \frac{3}{16}\varepsilon^2 A_1^5 z\right]\cos(3\omega t + 3\psi_0) \tag{40d}$$

and for six-term approximation

$$A_0 = A_2 = A_4 = 0, \tag{41a}$$

$$A_3 = -\tfrac{1}{32}\varepsilon A_1^3(9z-1) + \tfrac{3}{512}\varepsilon^2 A_1^5(63z^2 + 110z - 13), \tag{41b}$$

$$A_5 = \tfrac{1}{1024}\varepsilon^2 A_1^5(225z^2 - 34z + 1), \tag{41c}$$

$$\omega^2 = \omega_e^2[1 - \tfrac{3}{4}\varepsilon A_1^2(z-1) + \tfrac{3}{128}\varepsilon^2 A_1^4(33z^2 - 34z + 1)], \tag{41d}$$

$$\begin{aligned} v = {} & A_1\cos(\omega t + \psi_0) - [\tfrac{1}{32}\varepsilon A_1^3(9z-1) - \tfrac{3}{512}\varepsilon^2 A_1^5(63z^2 + 110z - 13)]\cos(3\omega t + 3\psi_0) \\ & + [\tfrac{1}{1024}\varepsilon^2 A_1^5(225z^2 - 34z + 1)]\cos(5\omega t + 5\psi_0). \end{aligned} \tag{41e}$$

These relationships are obtained under the assumptions that $A_3 \ll A_1$. A_1 and ψ_0 are found from initial conditions. Note that Eq. (41d) is the same as Eq. (37a) found by the Lindstedt method.

Obtaining the amplitudes and phase angles in both analytical solutions is quite difficult owing to highly nonlinear relations. Therefore, two codes in MATLAB for analytical solutions were written whose results will be presented along with pure numerical solutions in the next section.

4. Results and discussion

Two MATLAB codes were written to solve both Eqs. (21) and (30) by numerical integration, respectively. As is mentioned in the preceding section, two additional codes in MATLAB for analytical solutions were prepared since the amplitudes and phase angles in analytical solutions are difficult to obtain symbolically. In this section, the analytical and numerical results are presented in graphics and compared to each other. Furthermore, it will be studied how two non-dimensional parameters ε and ξ affect the system frequencies.

Before starting to present the numerical results a brief explanation appears necessary. The mass m is equal to unity for all solutions. For the nonlinear spring the ratio of the nonlinear part $\varepsilon k_2 v^3$ to the linear one $k_2 v$ is εv^2. If this ratio is close to or greater than 1 the nonlinearity becomes dominant. It is clear that this condition can be reached by choosing the initial deflection $v(0) = v_0$ larger than 1 even if ε is sufficiently small, e.g., is smaller than 1. For this reason the curves for v, u $(= y_1)$ and y_2 $(= u + v)$ were plotted for three different initial deflections for v: $v_0 < 1$, $v_0 = 1$ and

$v_0 > 1$. For each case the initial relative velocity $\dot{v}(0) = \dot{v}_0$ was taken to be zero. The condition of $\dot{v}_0 = 0$ does not guarantee that $\dot{y}_1(0) = \dot{y}_2(0) = 0$, but $\dot{y}_1(0) = \dot{y}_2(0)$. However it is assumed that the system is at rest initially. ε was set to 0.5 for Figs. 3–11, in order that the Lindstedt solution preserves its validity with regard to ε as a perturbation parameter. Additionally, three different values were assigned to the parameter $\xi = k_2/k_1$: 0.1, 1 and 10; hence nine separate combinations occur when considered along with initial deflections of three different orders. In what follows, the nine combinations mentioned above will be treated under three basic cases regarding ξ values and their subcases corresponding to different initial deflections, presenting the associated figures. Every figure includes three graphics, and in each of these graphics three curves which represent the numerical, the Lindstedt and the harmonic balance six-term solutions (henceforth, the NS, the LS and HB6S in short) only for that variable. Note that the solution curves associated with Eqs. (30),

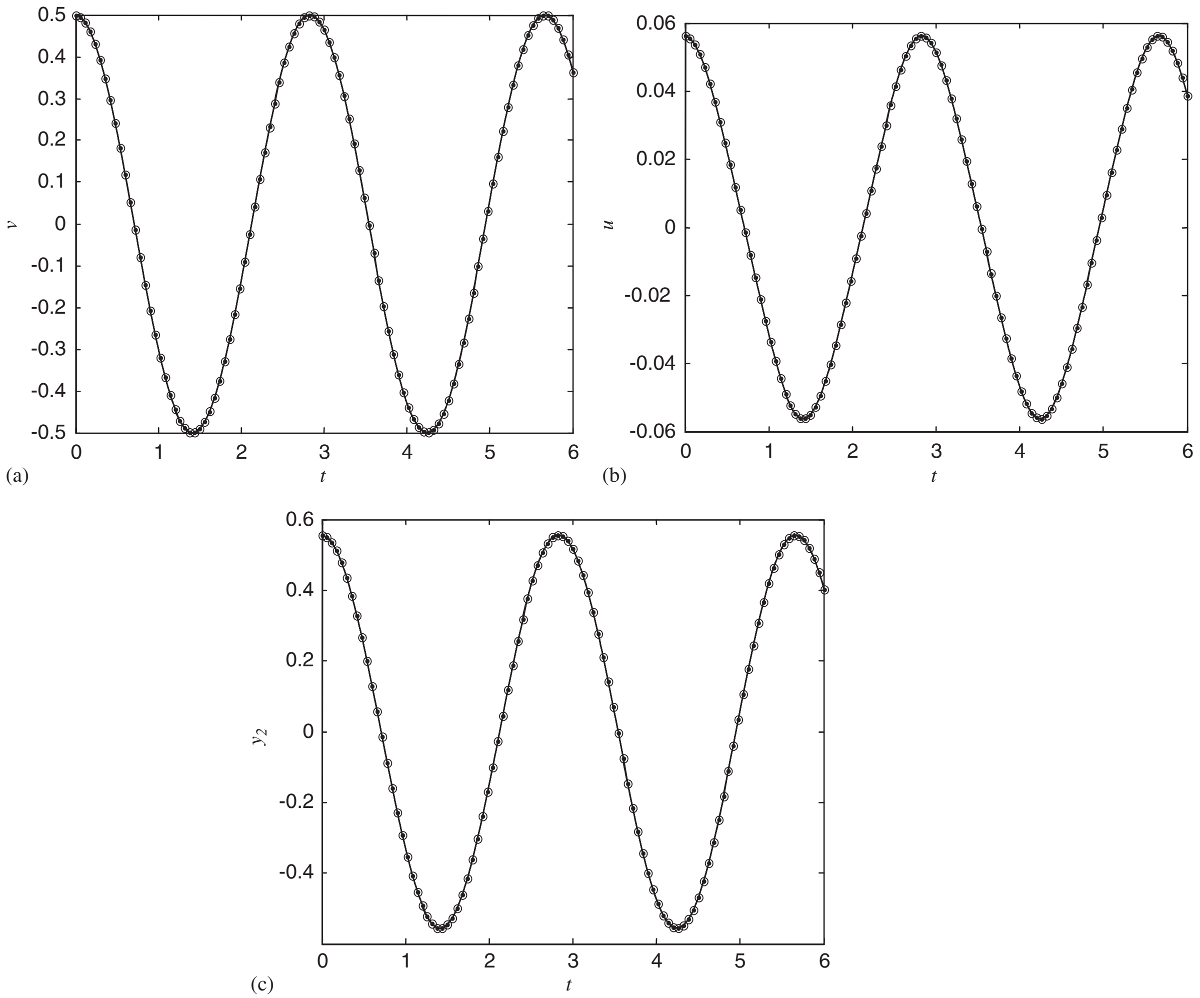

Fig. 3. Variations of spring deflections and mass displacement over time for the parameter values $\xi = 0.1$ ($k_1 = 50, k_2 = 5$), $v_0 = 0.5$. (a) Deflection of nonlinear spring, v (relative displacement), (b) deflection of linear spring, u, (c) displacement of mass, y_2. ——, NS; —●—, LS; —○—, HB6S.

i.e., the set of DAE, are not plotted in these graphics because they are observed always to be in complete agreement with the solution curves of Eq. (21), i.e., with the NS curves. The HB6S curves were preferred instead of the harmonic balance four-term (the solutions (the HB4S abbreviated)) since the former provide an overall good approximation.

Case 1: $\xi = 0.1$ ($\Leftarrow k_1 = 50, k_2 = 5$).

Subcase 1a: $v_0 = 0.5\,(<1)$. Fig. 3 is associated with this subcase. Under the given circumstances, the initial deflection of linear spring is $u_0 = 0.050625$ by Eq. (31) and that of the nonlinear one is 0.5. It is easily concluded that the nonlinear spring has a dominant effect on the frequency of free vibration due to linear spring being very hard, and hence, the system behaviour approaches to that of a system grounded only with the nonlinear spring. In such a case, numerical and analytical solutions are expected to almost completely coincide if the perturbation is small.

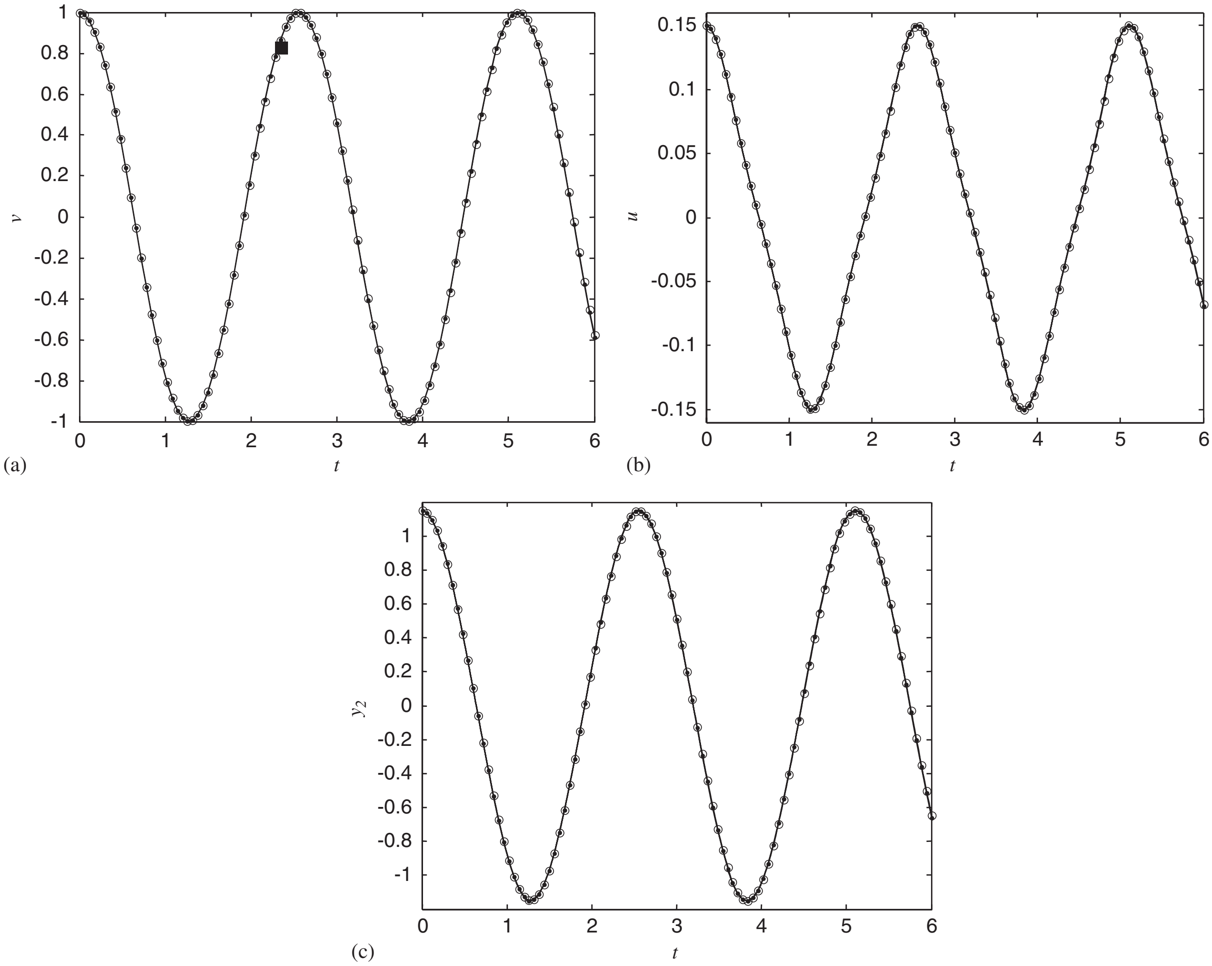

Fig. 4. Variations of spring deflections and mass displacement over time for the parameter values $\xi = 0.1$ ($k_1 = 50, k_2 = 5$), $v_0 = 1$. (a) Deflection of nonlinear spring, v (relative displacement), (b) deflection of linear spring, u, (c) displacement of mass, y_2. ——, NS; —●—, LS; —○—, HB6S.

It is noteworthy that the system will transform into a Duffing oscillator when k_1 goes to infinity. In Figs. 3a–c, the perfect coincidence of analytical and numerical solutions is remarkable. The system exhibits a motion resembling the ideal sinusoidal motion. However, it is not surprising because $\varepsilon v_0^2 = 0.0125 \ll 1$, that is, the nonlinearity is weak. Although it is not plotted in these figures the HB4S agrees with the others completely.

Subcase 1b: $v_0 = 1$. The graphics in Fig. 4 belong to this case. Now, $u_0 = 0.15$ while $v_0 = 1$. The linear spring can be viewed as a highly stiff wall, but the nonlinearity is relatively strong since $\varepsilon v_0^2 = 0.5 \cdot 1^2 = 0.5$. Therefore, one may still expect that numerical and analytical solutions be in very good agreement with each other. Figs. 4a–c confirm this expectation.

Subcase 1c: $v_0 = 2\ (>1)$. The plots related to this case are given in Fig. 5. Here, $u_0 = 0.6$ as $v_0 = 2$. The maximum deflection of linear spring is about 30% of that of the nonlinear. Furthermore, $\varepsilon v_0^2 = 0.5 \cdot 2^2 = 2$, hence the nonlinearity is strong. The analytical solution,

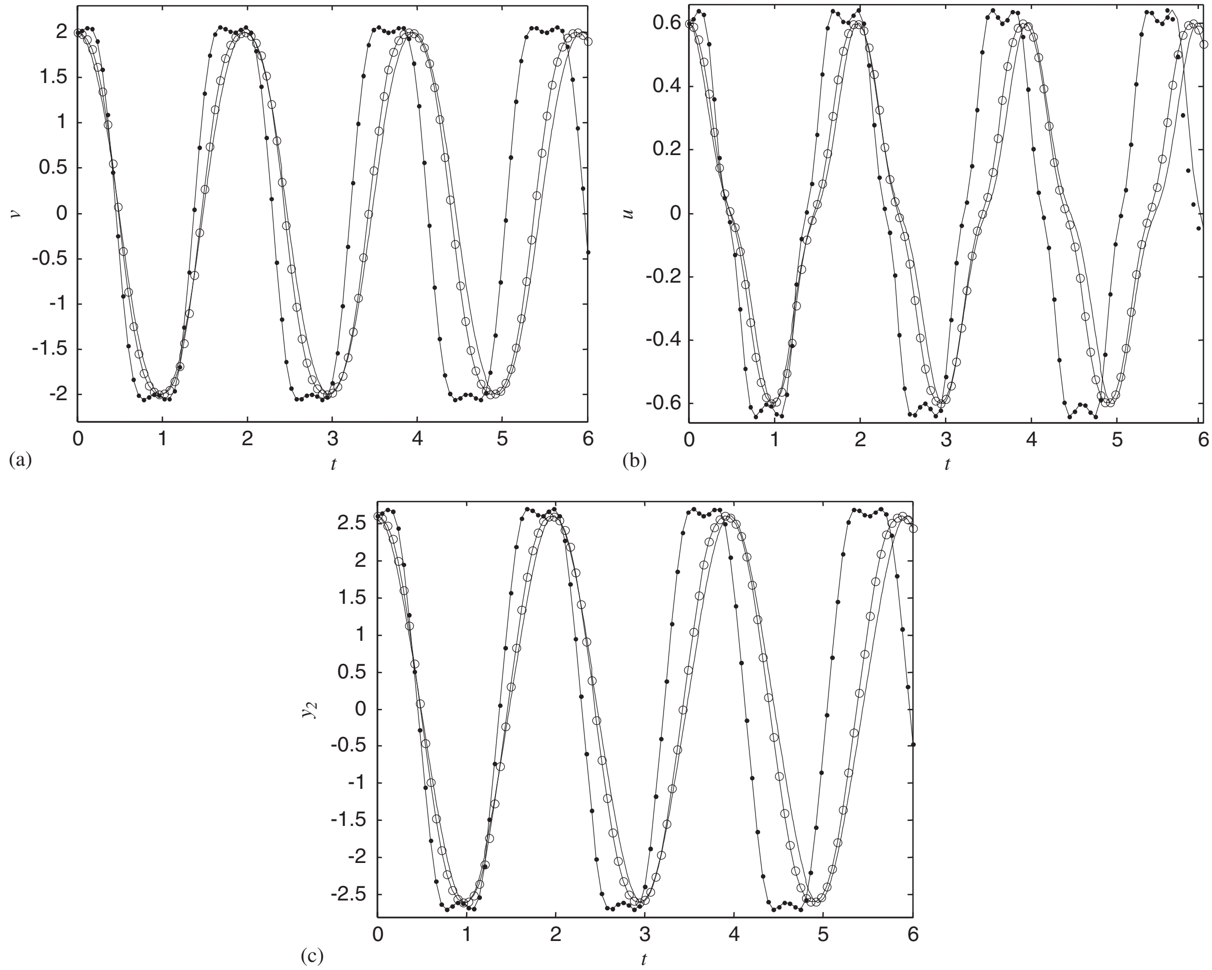

Fig. 5. Variations spring deflections and mass displacement over time for the parameter values $\xi = 0.1$ $(k_1 = 50, k_2 = 5), v_0 = 2$. (a) Deflection of nonlinear spring, v (relative displacement), (b) deflection of linear spring, u, (c) displacement of mass, y_2. ——, NS; —●—, LS; —○—, HB6S.

especially the LS could differ from the NS. The reason of emphasizing on the LS is that in this analytic solution the terms up to ε^2 were retained. For this solution to keep its validity more terms must be included in the series. The HB6S curves are observed to be more close to the NS in comparison with the LS. Since this method is not a perturbation method itself, a HB solution containing terms of sufficient number can represent the actual response of system. What is said so far is easily seen in Figs. 5a–c.

Case 2: $\xi = 1$ ($\Leftarrow k_1 = k_2 = 5$).

Subcase 2a: $v_0 = 0.5$ (<1). Here, $u_0 = 0.50625$ while $v_0 = 0.5$. The system is weak nonlinear because $\varepsilon v_0^2 = 0.0125$. Consequently, the system behaves as if it is grounded via two linear springs k_1 and k_2. Since the maximum spring deflections are of same order, and nonlinearity is weak, the analytical and numerical solutions are consistent with each other, Figs. 6a–c.

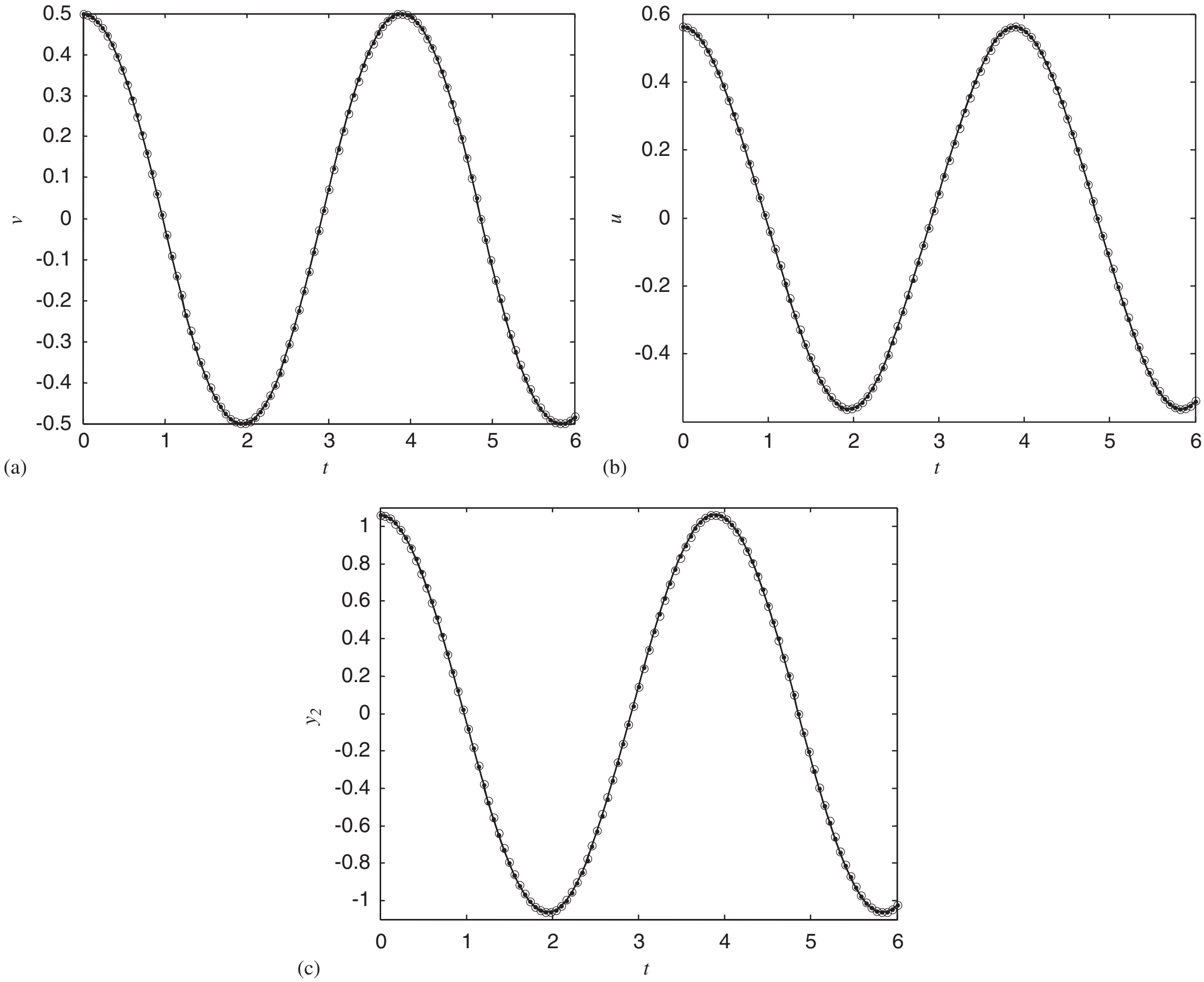

Fig. 6. Variations of spring deflections and mass displacement over time for the parameter values $\xi = 1$ ($k_1 = 5, k_2 = 5$), $v_0 = 0.5$. (a) Deflection of nonlinear spring, v (relative displacement), (b) deflection of linear spring, u, (c) displacement of mass, y_2. ——, NS; —●—, LS; —○—, HB6S.

Subcase 2b: $v_0 = 1$. In this case $u_0 = 1.5$. The nonlinearity has an effect that cannot be considered negligible. It is seen in Figs. 7a–c the solution curves slightly separate from each other except the points where they intersect the time axis. This means that the analytical solutions estimate the free vibration period very good, i.e., the LS and the HB6S prove to be satisfactory approximations under given conditions.

Subcase 2c: $v_0 = 2$ (>1). Here, $u_0 = 6$. As it is obvious from Figs. 8a–c the agreement among different solution curves is seriously lost. The nonlinearity is strong. The numbers of terms in both the LS and the HB6S are no more sufficient. The two series solution estimate the vibration period greater than it must be; hence a phase shift is observed.

Case 3: $\xi = 10$ $(\Leftarrow k_1 = 5, k_2 = 50)$.

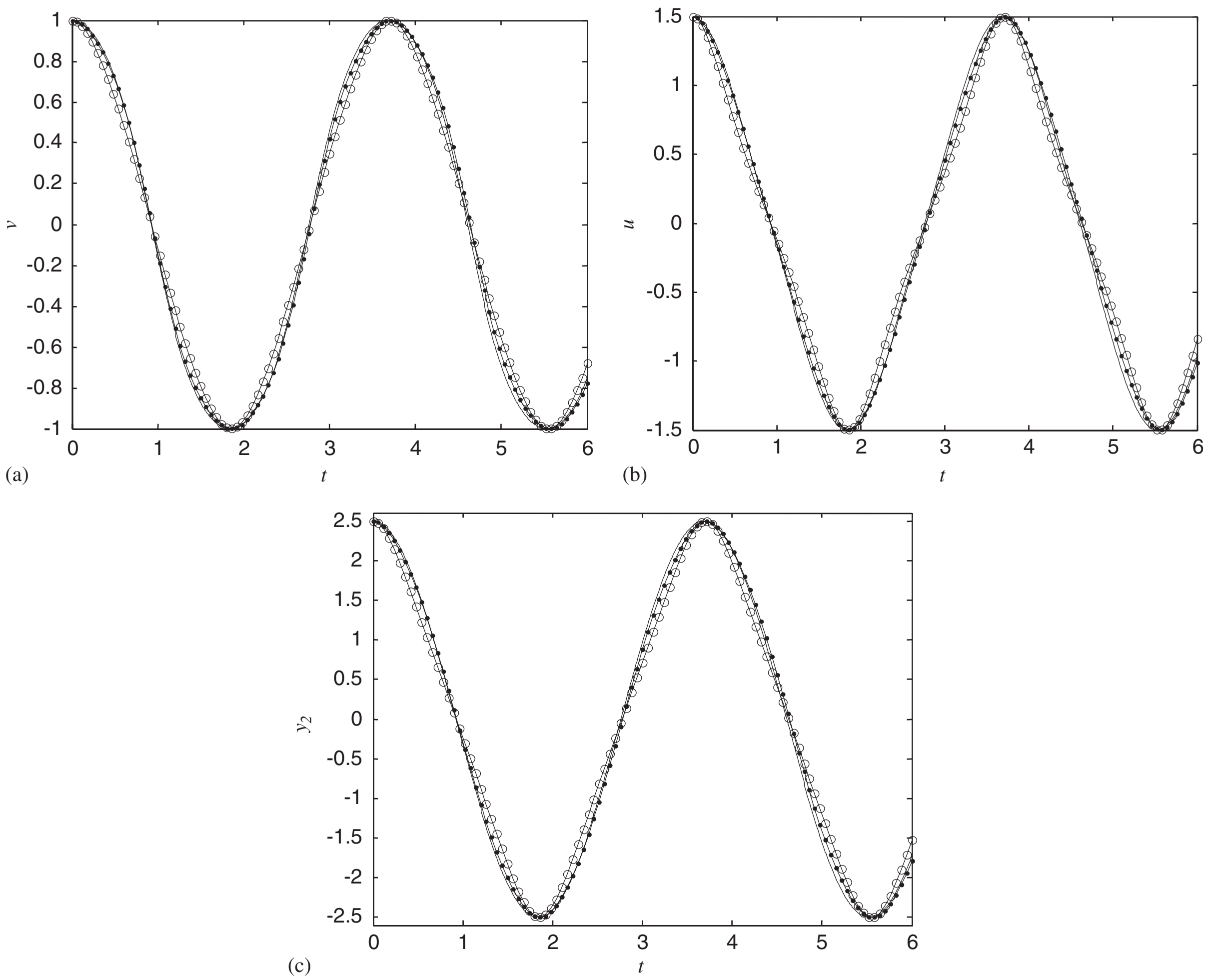

Fig. 7. Variations of spring deflections and mass displacement over time for the parameter values $\xi = 1$ $(k_1 = 5, k_2 = 5), v_0 = 1$. (a) Deflection of nonlinear spring, v (relative displacement), (b) deflection of linear spring, u, (c) displacement of mass, y_2. ——, NS; —●—, LS; —○—, HB6S.

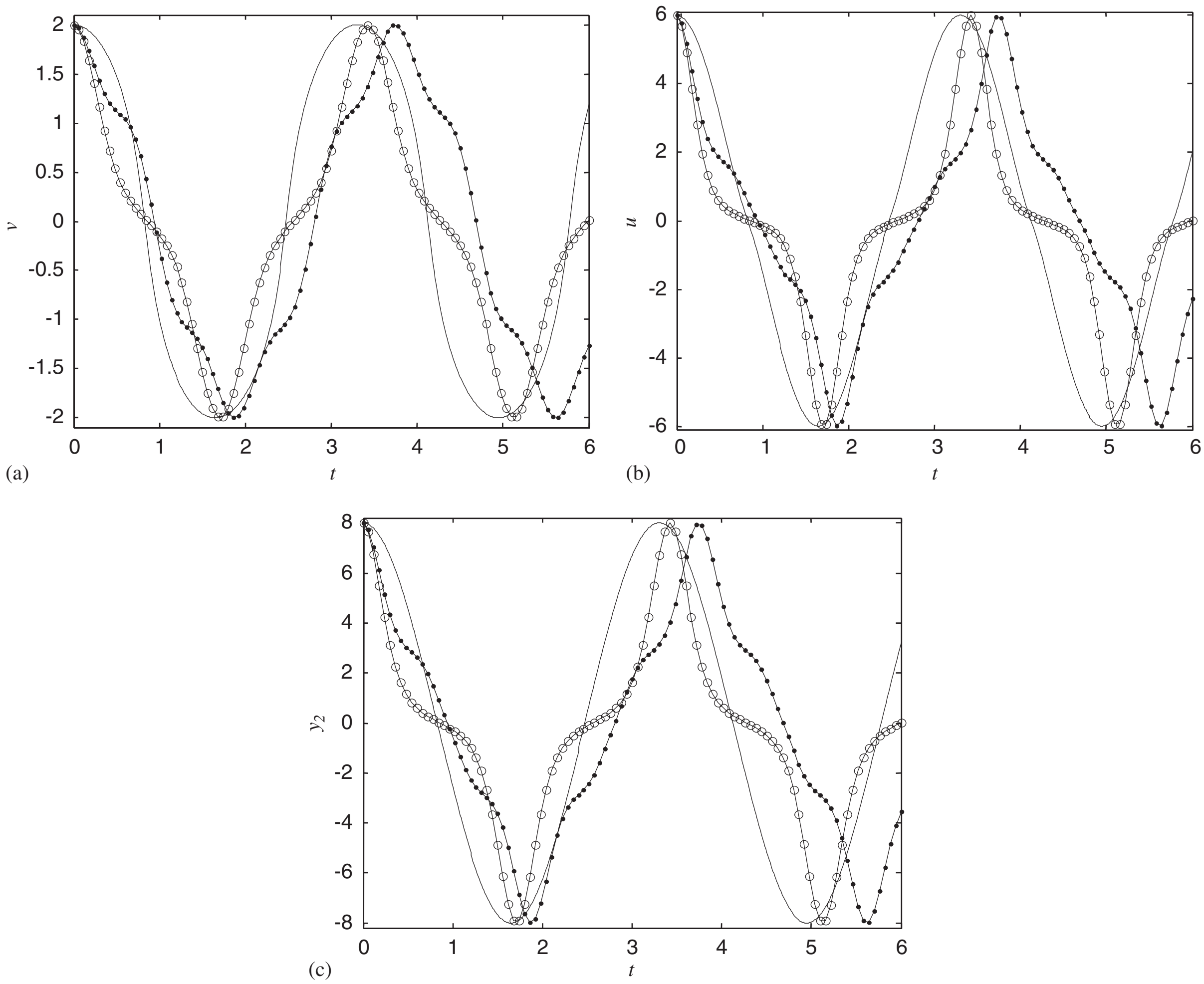

Fig. 8. Variations of spring deflections and mass displacement over time for the parameter values $\xi = 1$ $(k_1 = 5, k_2 = 5), v_0 = 2$. (a) Deflection of nonlinear spring, v (relative displacement), (b) deflection of linear spring, u, (c) displacement of mass, y_2. ——, NS; —●—, LS; —○—, HB6S.

Subcase 3a: $v_0 = 0.5$ (<1). Here, $u_0 = 5.0625$. The deflection of linear spring is ten times larger than that of the nonlinear. Therefore the time history of free vibration is determined mainly by the linear spring. The nonlinearity is weak. As a consequence of these facts all solution curves are in perfect agreement as seen in Figs. 9a–c.

Subcase 3b: $v_0 = 1$. In this case $u_0 = 15$. As is in the previous case the linear spring has a deflection quite larger than the others. There is moderate nonlinearity in the system. The solution curves vary similar to the subcase 2a as in Figs. 10a–c. The approximate solutions provide reliable period estimation although they are not in complete coincidence with the numerical.

Subcase 3c: $v_0 = 2$ (>1). Here, $u_0 = 60$. The nonlinearity is strong. However the linear spring has a position determining the attitude of the system. The deviation of the analytical solution

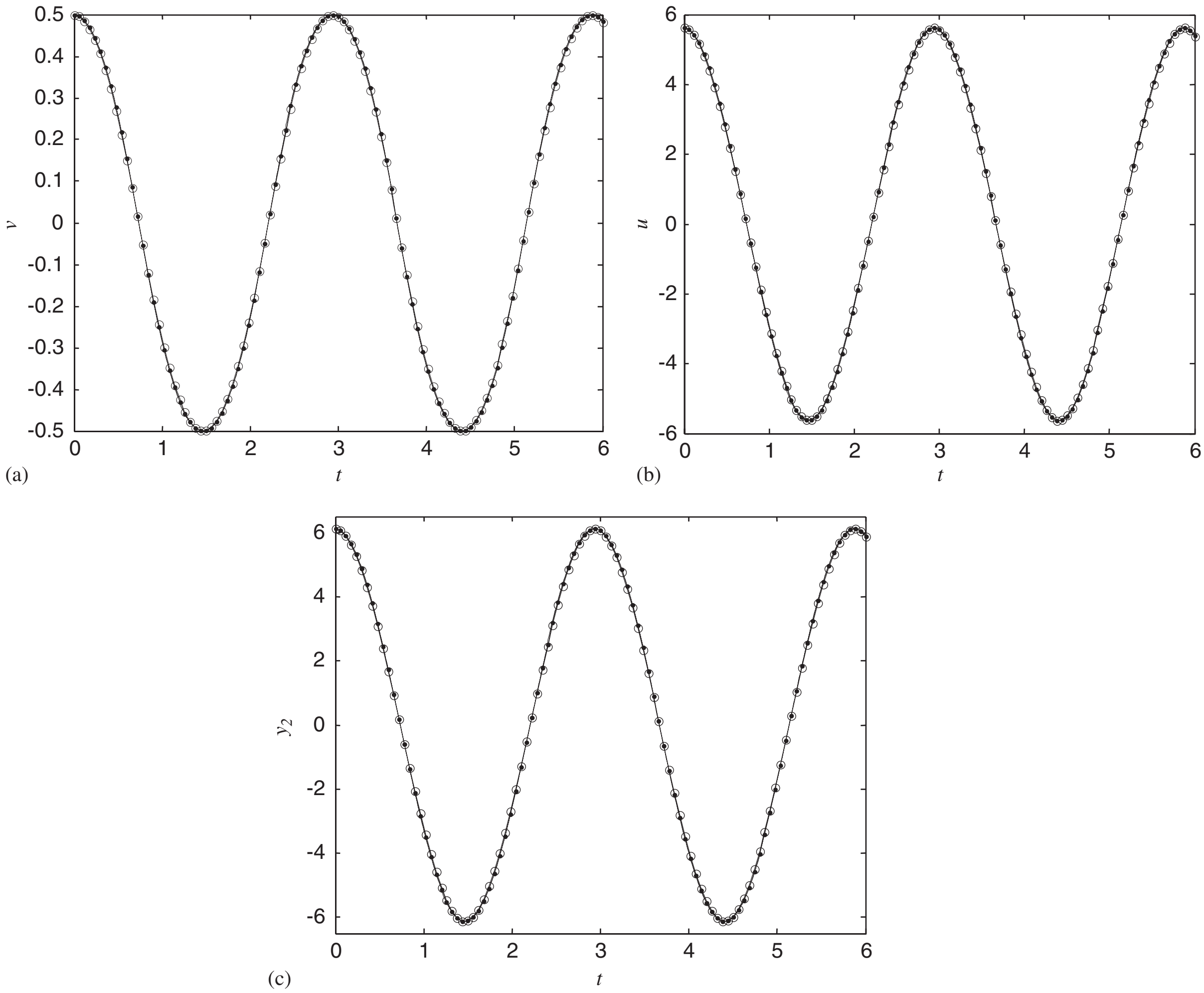

Fig. 9. Variations of spring deflections and mass displacement over time for the parameter values $\xi = 10$ ($k_1 = 5, k_2 = 50$), $v_0 = 0.5$. (a) Deflection of nonlinear spring, v (relative displacement), (b) deflection of linear spring, u, (c) displacement of mass, y_2. ——, NS; —●—, LS; —○—, HB6S.

curves from the numerical is very clear although the phase shift does not draw attention within first several periods of motion. For the intermediate values of deflections and mass displacement the analytical approximations will lead to erroneous results unless they are improved by increasing the number of terms included, Figs. 11a–c.

The HB4S and HB6S curves were observed to considerably differ from each other when $v_0 \geqslant 1$. For making a comparison between two series of different number of terms Figs. 12a–e were plotted for ξ and v_0 values chosen in this paper. For $\xi = 0.1$ both series are similar in shape but the HB4S has a smaller period, Fig. 12e. Therefore there occurs a shift between the curves.

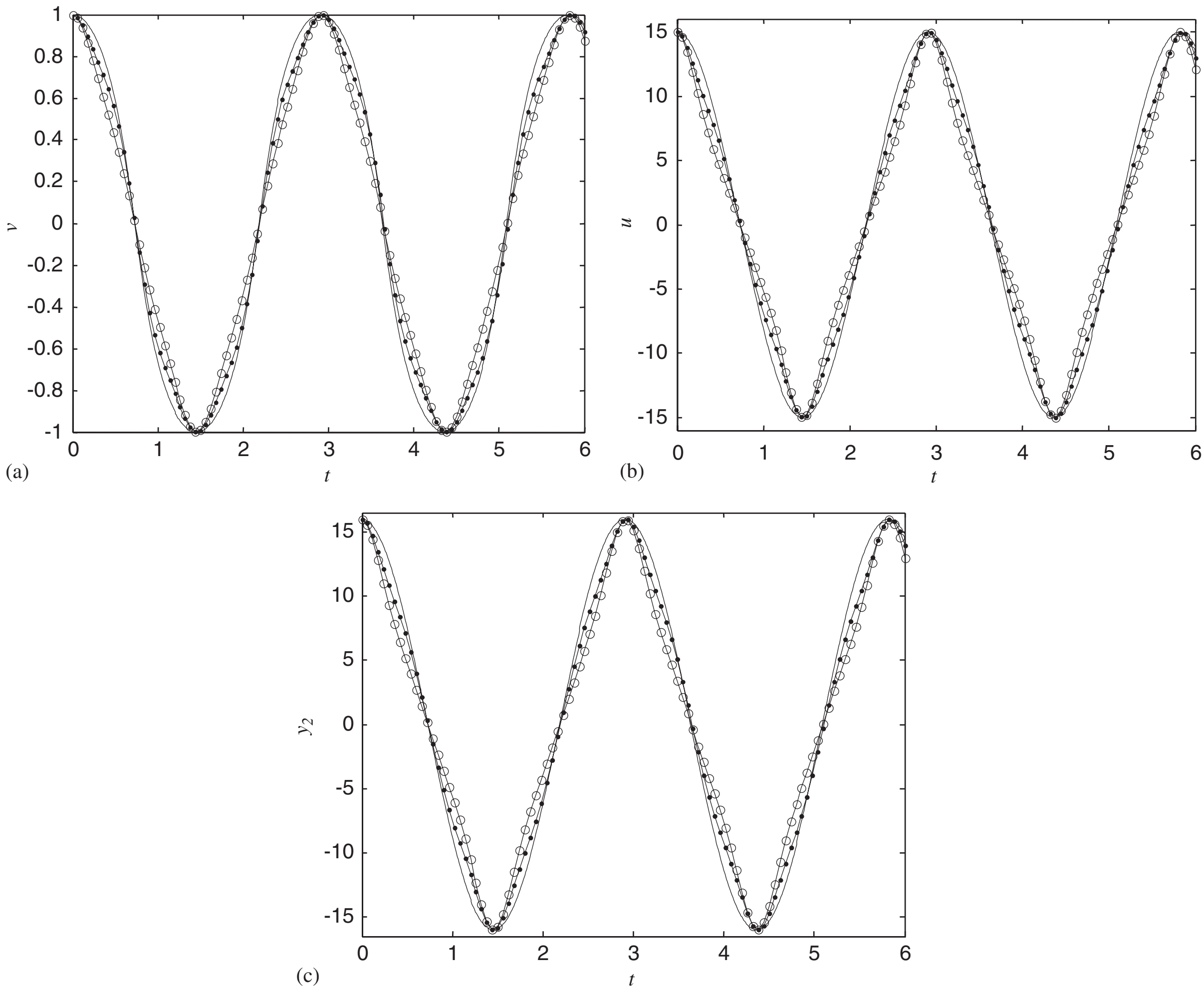

Fig. 10. Variations of spring deflections and mass displacement over time for the parameter values $\xi = 10$ ($k_1 = 5, k_2 = 50$), $v_0 = 1$. (a) Deflection of nonlinear spring, v (relative displacement), (b) deflection of linear spring, u, (c) displacement of mass, y_2. ——, NS; —●—, LS; —○—, HB6S.

Fig. 13 shows how the non-dimensional frequency of system changes with ε and ξ when $v_0 = 1$. It is noticeable that for the values $\xi \geqslant 1$ the ratio ω/ω_e remains almost constant such that even ε has no serious effect on the relative frequency. However, for $\xi < 1$ the influence of ε (i.e., the nonlinearity) becomes observable, especially for $|\varepsilon| = 0.5$.

In Fig. 14 the variation of the amplitude A_0 of the first term in the LS with respect to ε and ξ is plotted for $v_0 = 1$. In contrast with the frequency, ε affects the amplitude A_0 significantly for $\xi \geqslant 1$. For $0.5 \geqslant \xi \geqslant 0.1$ ε has practically no influence on the amplitude of the zero-order term. Finally, in Fig. 15 the variation of the amplitude A_0 with ε and ξ is depicted provided that $y_{20} = u_0 + v_0 = 1$. Here, the effect of ε on the amplitude is very weak while a decrease in ξ leads to increased amplitudes.

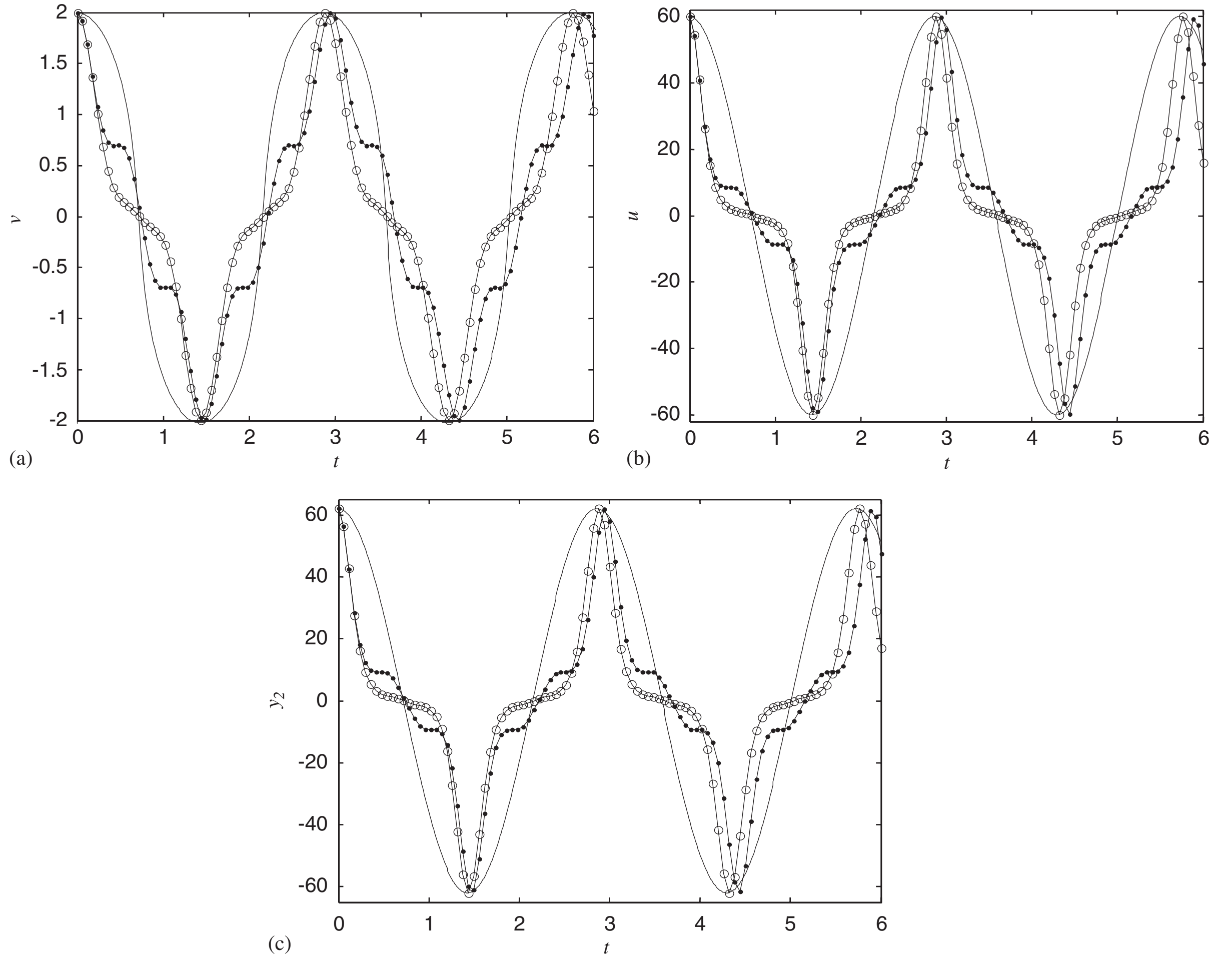

Fig. 11. Variations of spring deflections and mass displacement over time for the parameter values $\xi = 10$ $(k_1 = 5, k_2 = 50)$, $v_0 = 2$. (a) Deflection of nonlinear spring, v (relative displacement), (b) deflection of linear spring, u, (c) displacement of mass, y_2. ——, NS; —●—, LS; —○—, HB6S.

5. Conclusions

In this paper it is shown that the motion of a mass grounded via linear and nonlinear springs in series leads to a set of differential algebraic equations (DAE). However, introducing a suitable variable that represents the deflection of nonlinear spring, one can obtain a nonlinear ordinary differential equation (ODE). Once this equation is solved the original variables of problem are reached by inverse transformations. In this study it is assumed that the system is at rest initially. The set of DAE in new variables was solved using ode15s that is a built-in ODE-solver in MATLAB. The ODE was also solved by means of the same code. Beside the numerical solution of the ODE two analytical approximate solutions were developed. One of them is based on the well-known Lindstedt method that assures eliminating secular terms. For the other solution the

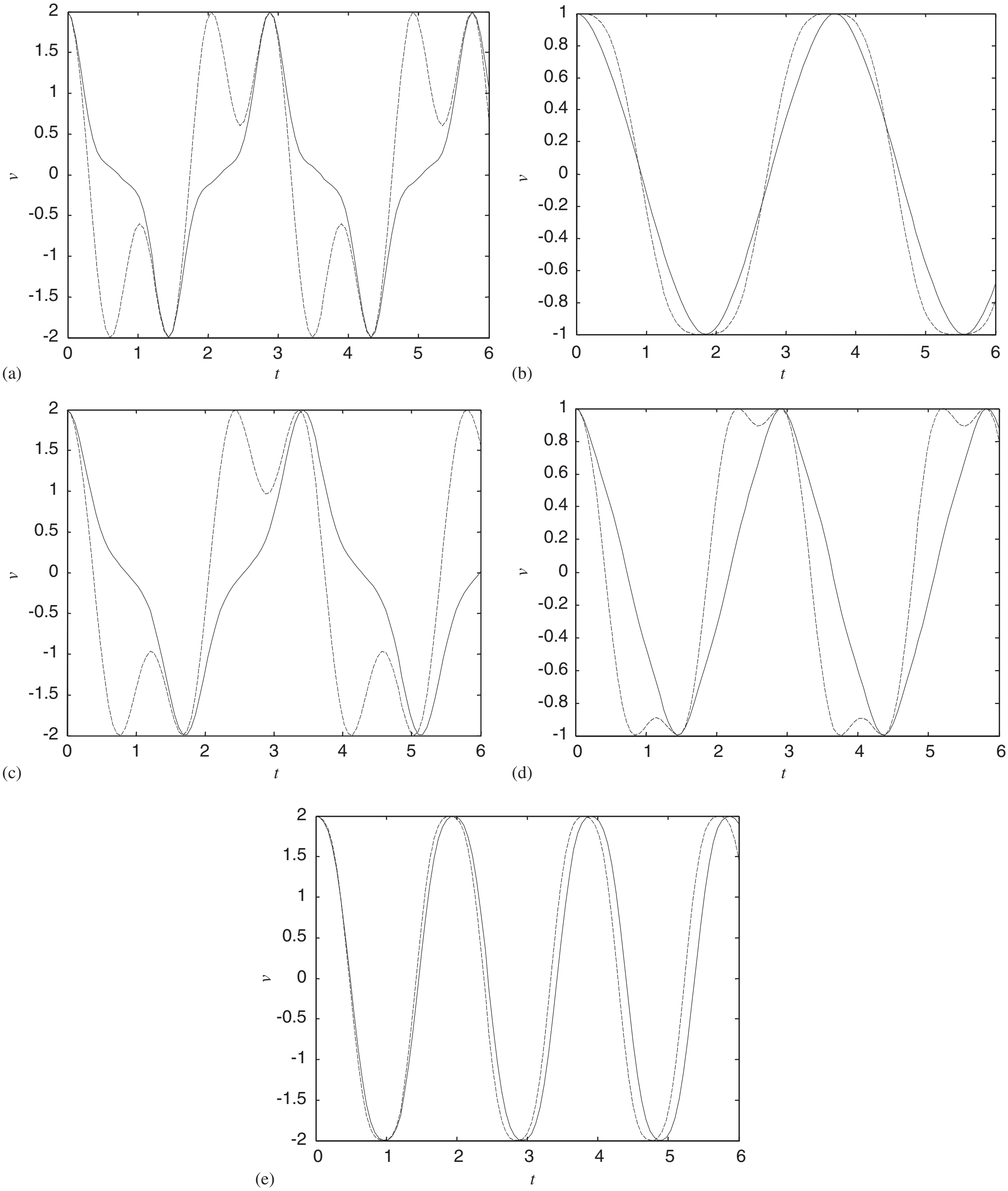

Fig. 12. Variation of relative displacement v (= deflection of nonlinear spring) according to two harmonic balance solutions with different term numbers for the parameter values (a) $\xi = 10,\ v_0 = 2$, (b) $\xi = 10,\ v_0 = 1$, (c) $\xi = 1,\ v_0 = 2$, (d) $\xi = 1,\ v_0 = 1$, (e) $\xi = 0.1,\ v_0 = 2$. - - - - - -, HB4S; ——, HB6S.

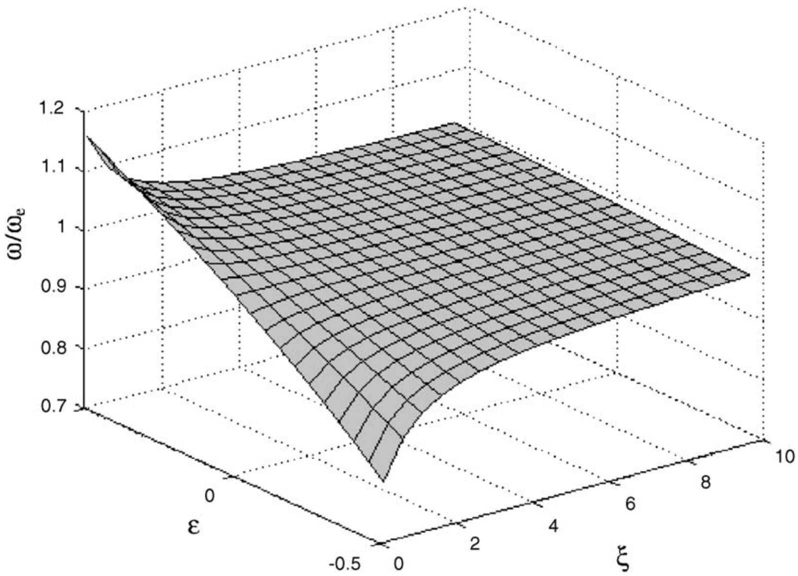

Fig. 13. Relative system frequency with respect to ε and ξ.

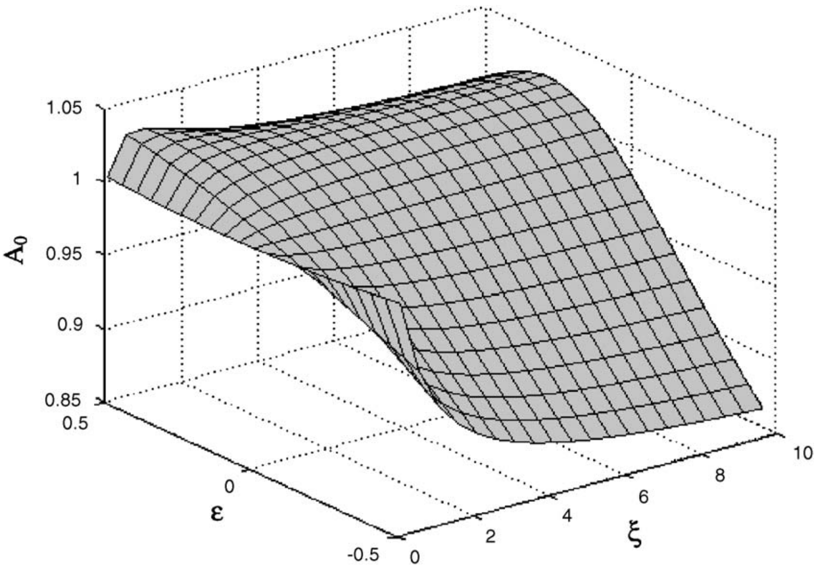

Fig. 14. Variation of amplitude A_0 of zeroth-order approximation in the Linstedt's method with ε and ξ under the assumption of a constant value of $v_0 = 1$.

harmonic balance method was employed. The Lindstedt solution includes the terms up to second order in ε while two other series developed using the HB method contain first four, and six terms, respectively, for comparison purposes.

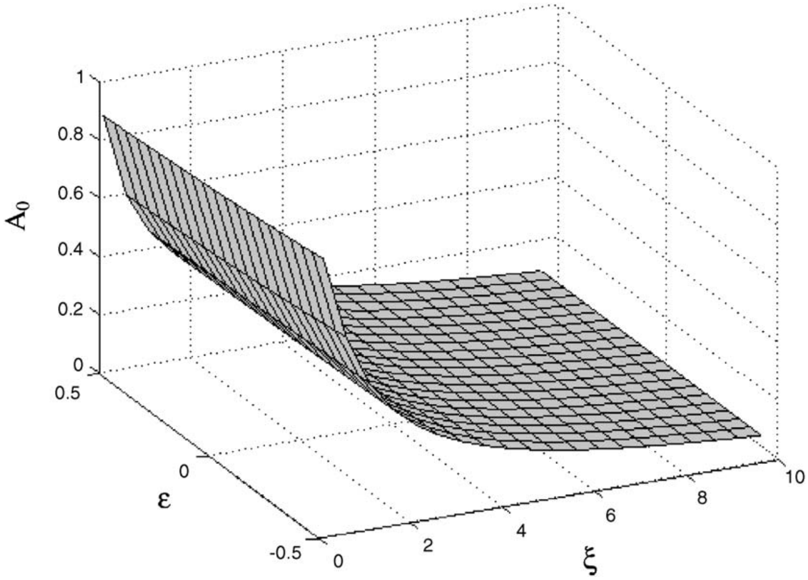

Fig. 15. Variation of amplitude A_0 of zeroth-order approximation in the Linstedt's method with ε and ξ under the assumption of a constant value of $y_{20} = 1$.

When Eqs. (37b) and (41e) are compared with each other it is realized that the coefficients (amplitudes) of the harmonics of third order are different in these two solutions whereas those of the harmonics of first and fifth order are completely same. Though it is not given here, it is observed that the correcting terms are added to the amplitudes of the harmonics of both third and fifth order when the number of terms in the HBM is increased to seven. It is obvious that the seventh-order harmonic also appears in this case. As a result, when taking the terms of adequate number, the HBM solutions are observed to be in full agreement with the Lindstedt solution, which would be expected since both approaches are based on the common assumption that a periodic solution exists although the ideas beyond both techniques are quite different.

Another noteworthy point is that both Eq. (37b) by the Lindstedt method and Eqs. (40d) and (41e) by the HBM, all of which are expressed in v, do not include the harmonics of even order. It is a well-known fact that this is the case for a single-degree-of-freedom system having a cubic nonlinear spring/restoring force characteristic as is explained in many standard textbooks on nonlinear vibrations [1,3]. However, that the solution for v has no harmonics of even order, does not mean that the absolute displacement $y_2 = u + v$ does not include even-order harmonics.

Although ε is considered as a perturbation indicator one easily notices from the characteristic of nonlinear spring that the initial deflection plays an important role in determining the range of validity of the Lindstedt solution because the assumption that the terms of higher order than zero have small contribution into the solution underlies this method. If this assumption is violated due to large deflections of nonlinear spring the LS must include terms of sufficient number to keep its validity. The HB solution is different from the LS regarding the ideas they are based on. From Figs. 3 to 11 it is concluded that the LS and HB6S in their present forms derived in this paper are satisfactory approximations for the system under study regarding both the period and the amplitude of motion provided that $v_0 \leqslant 1$ and $0.1 \leqslant \xi \leqslant 10$.

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